Optimisation of the lowest eigenvalue for surface δ-interactions and for Robin Laplacians

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in collaboration with
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The Neumann Laplacian: similar spectral inequality is trivial: $\lambda_1^{\mathrm{N}}(\Omega) = 0$.
Non-trivial for the Robin Laplacian and for δ-interactions on surfaces.

Part I. Schrödinger operators with δ-interactions on hypersurfaces

- P. Exner and V. L., *A spectral isoperimetric inequality for cones*, `arXiv:1512.01970`, 2015, *to appear in Lett. Math. Phys.*
- V. L., *Spectral isoperimetric inequalities for δ-interactions on open arcs and for the Robin Laplacian on planes with slits*, `arXiv:1609.07598`, 2016.

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The lowest spectral point for H_α^Σ

$\mu_1^\alpha(\Sigma) := \inf \sigma(\mathsf{H}_\alpha^\Sigma)$.

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(i) H_α^Σ models a 'leaky' quantum system wherein a particle is confined to Σ but the tunnelling between different parts of Σ is not neglected.

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Motivation from spectral geometry

Characterise the spectrum of H_α^Σ in terms of Σ!

This question can be asked for various shapes of Σ

(i) Exner-Kovařík-15 and the references therein.

(ii) Universal description of spectrum for general Σ can hardly be found!

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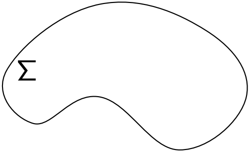

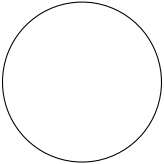

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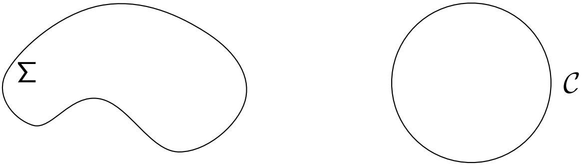

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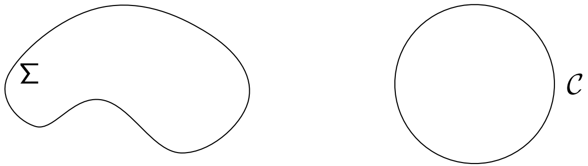

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Theorem (Exner-05, Exner-Harrell-Loss-06)

$$\begin{cases} |\Sigma| = |\mathcal{C}| \\ \Sigma \not\cong \mathcal{C} \end{cases} \quad \Longrightarrow \quad \mu_1^\alpha(\mathcal{C}) > \mu_1^\alpha(\Sigma), \quad \forall\, \alpha > 0.$$

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1) Birman-Schwinger principle; 2) the line segment is the shortest path connecting two fixed points; 3) strict monotonous decay of $K_0(\cdot)$.

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Open question

The shape of the optimizer under two constraints simultaneously:

1) fixed endpoints $P, Q \in \mathbb{R}^2$;

2) fixed length $L \in (|P - Q|, \infty)$?

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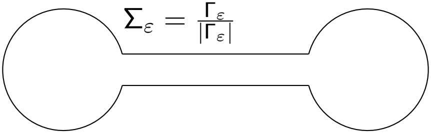

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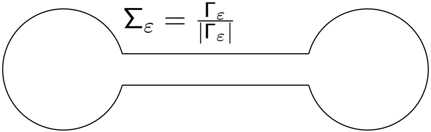

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For all $\alpha > 0$ and all sufficiently small $\varepsilon > 0$ holds $\sigma_{\rm d}(\mathsf{H}_\alpha^{\Sigma_\varepsilon}) = \varnothing$.

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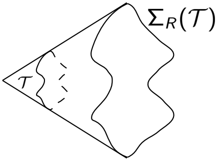

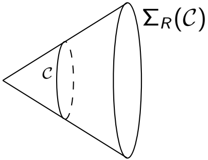

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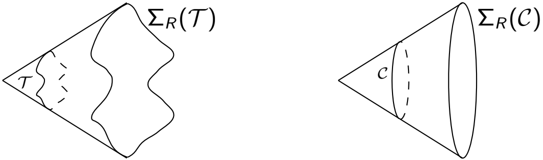

Theorem (Exner-V.L.-15)

$$\begin{cases} |\mathcal{C}| = |\mathcal{T}| \\ \mathcal{C} \not\cong \mathcal{T} \end{cases} \implies \mu_1^\alpha(\Sigma_R(\mathcal{C})) > \mu_1^\alpha(\Sigma_R(\mathcal{T})), \quad \forall\, \alpha \geq \alpha_*(\Sigma_R(\mathcal{C})).$$

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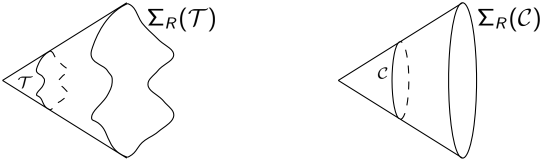

Theorem (Exner-V.L.-15)

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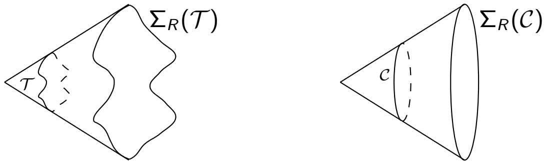

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The borderline case $\alpha = \alpha_*(\Sigma_R(\mathcal{C}))$: $\mu_1^\alpha(\Sigma_R(\mathcal{C})) = 0$ and $\mu_1^\alpha(\Sigma_R(\mathcal{T})) < 0$.

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Passing in the result for truncated cones to the limit $R \to +\infty$.

Part II. The Robin Laplacian

- D. Krejčiřík and V. L., *Optimisation of the lowest Robin eigenvalue in the exterior of a compact set*, `arXiv:1608.04896`, 2016.
- V. L., *Spectral isoperimetric inequalities for δ-interactions on open arcs and for the Robin Laplacian on planes with slits* , `arXiv:1609.07598`, 2016.

Definition of the Robin Laplacian

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$H^1(\Omega) \ni u \mapsto \mathfrak{h}_\beta^\Omega[u] := \|\nabla u\|^2_{L^2(\Omega;\mathbb{C}^d)} + \beta \|u|_{\partial\Omega}\|^2_{L^2(\partial\Omega)}$ for $\beta \in \mathbb{R}$.

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The lowest spectral point for H_β^Ω

$\nu_1^\beta(\Omega) := \inf \sigma(\mathsf{H}_\beta^\Omega)$.

Motivations

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- Methods and results are frequently very different from δ-interactions.
- Although, the spectral problems have much in common.

The Robin Laplacian on a bounded domain

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Theorem (Bossel-86 ($d = 2$), Daners-06 ($d \geq 3$))

$$\begin{cases} |\Omega| = |\mathcal{B}| \\ \Omega \not\cong \mathcal{B} \end{cases} \quad \Longrightarrow \quad \nu_1^\beta(\mathcal{B}) < \nu_1^\beta(\Omega), \quad \forall\, \beta > 0.$$

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Open questions: generalisations of the last theorem for $d \geq 3$ and for $d = 2$ under the constraint $|\Omega| = |\mathcal{B}|$ for simply connected domains.

The Robin Laplacian on exterior domains: fixed perimeter

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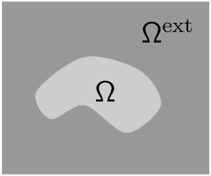
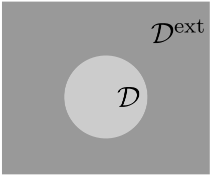
$\Omega \subset \mathbb{R}^2$ – bounded, simply connected, C^∞-smooth. $\Omega^{\rm ext} := \mathbb{R}^2 \setminus \overline{\Omega}$.





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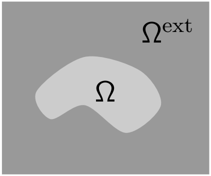

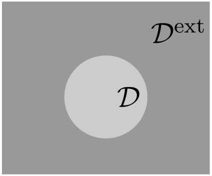

Theorem (Krejčiřík-VL-16, $d = 2$)

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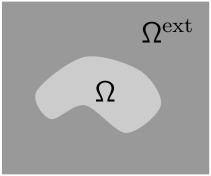

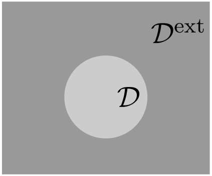

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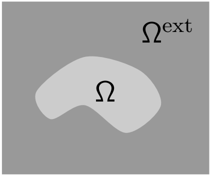

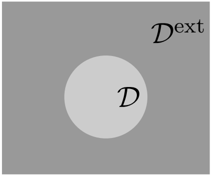

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Non-convex case: joint work in progress with D. Krejčiřík.

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Lemma

$\mathcal{D}_R$ – *a disc of radius* R. $\beta < 0$. $\Longrightarrow$ $R \mapsto \nu_1^\beta(\mathcal{D}_R^{\rm ext})$ *is strictly decaying.*

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Strong coupling (Kovařík-Pankrashkin-16)

$\nu_1^\beta(\Omega_r^{\mathrm{ext}}) - \nu_1^\beta(\mathcal{D}_R^{\mathrm{ext}}) = \beta\left(\frac{1}{R} - \frac{1}{r}\right) + o(\beta)$ as $\beta \to -\infty$.

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For all $\beta < 0$ with $|\beta|$ large enough

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For sufficiently small r and all $\beta < 0$ with $|\beta|$ large enough

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The Robin Laplacian on a plane with a slit

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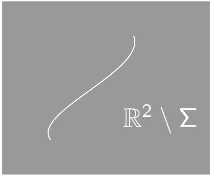

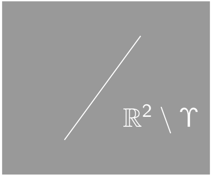

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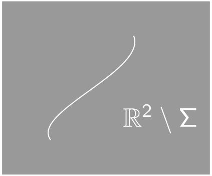

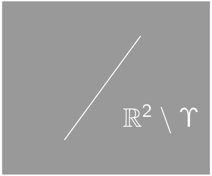

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$$\begin{cases} |\Sigma| = |\Upsilon| \\ \Sigma \not\cong \Upsilon \end{cases} \quad \Longrightarrow \quad \nu_1^\beta(\mathbb{R}^2 \setminus \Upsilon) > \nu_1^\beta(\mathbb{R}^2 \setminus \Sigma), \quad \forall \beta < 0.$$

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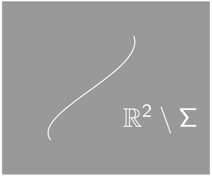

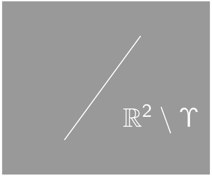

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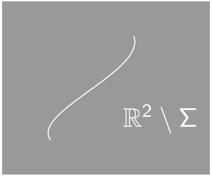

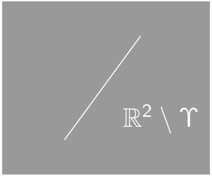

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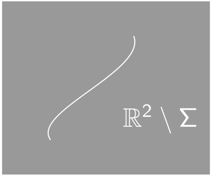

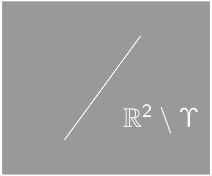

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- The claim follows from $\mu_1^{2\beta}(\Sigma) < \mu_1^{2\beta}(\Upsilon)$. □

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Thank you

Thank you for your attention!