On the bound state induced by δ-interaction on a weakly deformed plane

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joint work with

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Locally deformed hyperplane

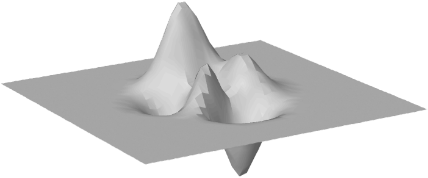
$\Pi \subset \mathbb{R}^3$ – a hyperplane.

Locally deformed hyperplane

Lip. surface $\Sigma \subset \mathbb{R}^3$ such that $\Sigma \setminus \mathcal{K} = \Pi \setminus \mathcal{K}$ for a compact set $\mathcal{K} \subset \mathbb{R}^3$.

Special locally deformed hyperplane

$\Sigma = \{(x, f(x)) \colon x \in \mathbb{R}^2\}$, where Lip. $f \colon \mathbb{R}^2 \to \mathbb{R}$ is compactly supported.



Avoiding handlebodies and other topological complications!

$f(\cdot)$ – profile function.

δ-interaction on a locally deformed plane

$\Sigma \subset \mathbb{R}^3$ – locally deformed hyperplane. $\alpha > 0$ – the coupling constant.

Quadratic form for $-\Delta - \alpha\delta_\Sigma$

$$Q_{\alpha,\Sigma}[u] = \int_{\mathbb{R}^3} |\nabla u|^2 - \alpha \int_\Sigma |u|^2, \qquad \mathrm{dom}\, Q_{\alpha,\Sigma} = H^1(\mathbb{R}^3).$$

Schrödinger operator with δ-interaction supported on Σ

$Q_{\alpha,\Sigma} \xrightarrow{1^{st}\text{-repr.}} \mathsf{H}_{\alpha,\Sigma}$ self-adjoint in $L^2(\mathbb{R}^3)$.

In physics

$\mathsf{H}_{\alpha,\Sigma}$ models a 'leaky' quantum system; a charged particle is confined to Σ but the tunnelling between different parts of Σ is not neglected.

Essential spectrum

Proposition

$\sigma_{\rm ess}(\mathsf{H}_{\alpha,\Sigma}) = [-\frac{1}{4}\alpha^2, +\infty)$.

- Separation of variables $\Rightarrow \sigma(\mathsf{H}_{\alpha,\Pi}) = \sigma_{\rm ess}(\mathsf{H}_{\alpha,\Pi}) = [-\frac{1}{4}\alpha^2, \infty)$.
- $(\mathsf{H}_{\alpha,\Sigma} - z)^{-1} - (\mathsf{H}_{\alpha,\Pi} - z)^{-1}$ is a compact operator for any $z \in \mathbb{C} \setminus \mathbb{R}$.
- Stability of essential spectrum $\Rightarrow \sigma_{\rm ess}(\mathsf{H}_{\alpha,\Sigma}) = \sigma_{\rm ess}(\mathsf{H}_{\alpha,\Pi}) = [-\frac{1}{4}\alpha^2, \infty)$

Geometrically induced bound states

Open problem *(Exner-08)*

$\sigma_{\rm d}(\mathsf{H}_{\alpha,\Sigma}) \neq \varnothing$ for any $\Sigma \neq \Pi$ and all $\alpha > 0$?

The lowest spectral point of $\mathsf{H}_{\alpha,\Sigma}$

$$\lambda_1^\alpha(\Sigma) = \inf_{\substack{u \in H^1(\mathbb{R}^3) \\ u \neq 0}} \frac{Q_{\alpha,\Sigma}[u]}{\|u\|^2_{L^2(\mathbb{R}^3)}} = \inf \sigma(\mathsf{H}_{\alpha,\Sigma})\,.$$

It suffices to show that $\lambda_1^\alpha(\Sigma) < -\frac{1}{4}\alpha^2$.

For the Dirichlet Laplacian on a layer build over Σ: $\sigma_{\rm d} \neq \varnothing$ if $\int_\Sigma \mathcal{K} \leq 0$. (*Duclos-Exner-Krejčiřík-01, Carron-Exner-Krejčiřík-04, Lin-Lu-07*).

The method applied for layers **fails** for the δ-interaction

Parallel coordinates are globally ill-defined and flattening Σ is not possible.

Large coupling behaviour

Assume that Σ is C^4-smooth and let κ_1, κ_2 be its principal curvatures.

$\kappa_1 \neq \kappa_2$ for $\Sigma \neq \Pi$.

Laplace-Beltrami operator on Σ

$-\Delta_\Sigma$ – self-adjoint in $L^2(\Sigma)$.

Min-max with a properly chosen test function

$\mu_1 := \inf \sigma\big(-\Delta_\Sigma - \frac{1}{4}(\kappa_1 - \kappa_2)^2\big) < 0$ if $\Sigma \neq \Pi$.

Large coupling asymptotics for $\Sigma \neq \Pi$ *(Exner-Kondej-03)*

$\lambda_1^\alpha(\Sigma) = -\frac{1}{4}\alpha^2 + \mu_1 + \mathcal{O}(\alpha^{-1} \log \alpha)$ as $\alpha \to +\infty$.

The answer is positive for all α large enough!

Weak local deformation setting

$\alpha > 0$ – fixed!

Profile function

C^2-smooth, compactly supported $f\colon \mathbb{R}^2 \to \mathbb{R}$ with a Lip. constant $\mathcal{L}_f > 0$.

The graph of $x \mapsto \beta f(x)$

$\Sigma_\beta(f) = \{(x, \beta f(x))\colon x \in \mathbb{R}^2\}$ for $\beta \geq 0$.

$\mathsf{H}_{\alpha,\beta} := \mathsf{H}_{\alpha,\Sigma_\beta(f)}$ and $\lambda_1^\alpha(\beta) := \lambda_1^\alpha(\Sigma_\beta(f))$ – shorthand notation.

Proposition (*Exner-Kondej-VL-17*)

$\lambda_1^\alpha(\beta) \geq -\frac{\alpha^2}{4}(1 + \beta^2\mathcal{L}_f^2)$ *and thus* $\lambda_1^\alpha(\beta) \to -\frac{\alpha^2}{4}$ *as* $\beta \to 0^+$.

Question

The exact behaviour of $\lambda_1^\alpha(\beta)$ in the limit $\beta \to 0^+$?

Main result

$\widehat{f} :=$ the Fourier transform of f.

$$\mathcal{D}_{\alpha,f} := \int_{\mathbb{R}^2} |p|^2 |\widehat{f}(p)|^2 \left(\alpha^2 - \frac{2\alpha^3}{\sqrt{4|p|^2+\alpha^2}+\alpha} \right) \mathrm{d}p > 0\,.$$

Theorem (*Exner-Kondej-VL-17*)

(i) $\#\sigma_{\rm d}(\mathsf{H}_{\alpha,\beta}) = 1$ *for all sufficiently small* $\beta > 0$.

(ii) *The lowest eigenvalue behaves as*

$$\lambda_1^\alpha(\beta) = -\frac{1}{4}\alpha^2 - \exp\left(-\frac{16\pi}{\mathcal{D}_{\alpha,f}\beta^2} \right)(1 + o(1))\,, \qquad \beta \to 0^+\,.$$

Resembles the behaviour of the $1^{\rm st}$-eigenvalue as $\varepsilon \to 0^+$ for $-\Delta - \varepsilon V$ in $\mathbb{R}^2$ with $V \in C_0^\infty(\mathbb{R}^2)$, $V \ge 0$ (*Simon-76*).

Birman-Schwinger principle in the flat metric

Green's function in $\mathbb{R}^3$

$G_\kappa(x) = \frac{e^{-\kappa|x|}}{4\pi|x|}$, $\kappa > 0$.

Surface measure on $\Sigma_\beta(f)$

$\mathrm{d}\sigma(x) = g_\beta(x)\mathrm{d}x$, where $g_\beta(x) = \left(1 + \beta^2|\nabla f(x)|^2\right)^{1/2}$.

Selfadj. BS-operator $\mathsf{Q}_\beta(\kappa)\colon L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$, $\kappa > 0$ ($\approx$ Weyl function)

$$(\mathsf{Q}_\beta(\kappa)\psi)(x) := \int_{\mathbb{R}^2} g_\beta(x)^{1/2} G_\kappa\big((x, \beta f(x)) - (y, \beta f(y))\big) g_\beta(y)^{1/2}\psi(y)\mathrm{d}y.$$

1st BS-principle (*Brasche-Exner-Kuperin-Šeba-94, Behrndt-Langer-VL-13*)

$$\forall\, \kappa > 0, \qquad \dim\ker\left(\mathsf{H}_{\alpha,\beta} + \kappa^2\right) = \dim\ker\left(\mathsf{I} - \alpha\mathsf{Q}_\beta(\kappa)\right).$$

Original formulation in $L^2(\Sigma_\beta(f))$ is inconvenient because $\Sigma_\beta(f)$ is varying.

Perturbative reformulation of the BS-principle

Recall that $\Sigma_\beta(f)$ is a local perturbation of Π.

Again apply BS-principle, now in $L^2(\mathbb{R}^2)$; unperturbed operator $\alpha \mathsf{Q}_0(\kappa)$.

$\delta := \sqrt{\kappa^2 - \frac{1}{4}\alpha^2} > 0$ for $\kappa > \frac{1}{2}\alpha$.

$\mathsf{D}_\beta(\delta) := \mathsf{Q}_\beta(\kappa) - \mathsf{Q}_0(\kappa)$ and $\mathsf{B}_\alpha(\delta) := (\mathsf{I} - \alpha \mathsf{Q}_0(\kappa))^{-1}$

2^{nd} BS-principle ($\forall\, \kappa > \frac{1}{2}\alpha$)

$$\begin{aligned}\dim\ker\left(\mathsf{H}_{\alpha,\beta} + \kappa^2\right) &= \dim\ker\left(\mathsf{I} - \alpha \mathsf{Q}_\beta(\kappa)\right)\\ &= \dim\ker\Big(\left(\mathsf{I} - \alpha \mathsf{Q}_0(\kappa)\right)\left(\mathsf{I} - \alpha \mathsf{B}_\alpha(\delta)\mathsf{D}_\beta(\delta)\right)\Big)\\ &= \dim\ker\Big[\mathsf{I} - \alpha \mathsf{B}_\alpha(\delta)\mathsf{D}_\beta(\delta)\Big].\end{aligned}$$

Implicit scalar equation on the lowest eigenvalue

For small $\beta > 0$ and $\kappa > \frac{1}{2}\alpha$

$$\dim\ker\left(\mathsf{H}_{\alpha,\beta} + \kappa^2\right) = \dim\ker\left[\mathsf{I} - \alpha\mathsf{B}_\alpha(\delta)\mathsf{D}_\beta(\delta)\right]$$
$$= \dots\dots\dots = \dim\ker[\mathsf{I} - \mathsf{P}_{\alpha,\beta}(\delta)],$$

with an "explicitly" given rank-one $\mathsf{P}_{\alpha,\beta}(\delta)$: $L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$.

For $\beta > 0$ small

$-\frac{1}{4}\alpha^2 - \delta^2$ is a simple eigenvalue of $\mathsf{H}_{\alpha,\beta}$ $\iff$ $\boxed{\operatorname{Tr}\mathsf{P}_{\alpha,\beta}(\delta) = 1}$.

We show existence & uniqueness of the solution for $\operatorname{Tr}\mathsf{P}_{\alpha,\beta}(\delta) = 1$ for all $\beta > 0$ small.

Representation for relativistic Schrödinger operator (*Ichinose-Tsuchida-93*) $\Rightarrow$

$$\operatorname{Tr}\mathsf{P}_{\alpha,\beta}(\delta) = -\frac{\beta^2 \ln\delta}{8\pi}\left(\mathcal{D}_{\alpha,f} + o_{\mathrm{u}}(1)\right), \qquad \beta \to 0^+. \quad \square$$

Related settings

The technique applies for mild non-local perturbations

Almost flat cones ($f(x) = \gamma(x_1^2 + x_2^2)^{1/2}$, $\gamma \to 0^+$) modification needed (formally $\mathcal{D}_{\alpha,f} = \infty$); *Ourmières-Bonafos–Pankrashkin–Pizzichillo-17*: $\gamma \to \infty$.

Similar analysis can be performed for space dimensions $d \geq 4$

We expect that $\sigma_{\rm d} = \varnothing$, $\forall\, \beta > 0$ sufficiently small.

Robin Laplacian in a locally perturbed half-space

(i) Existence of a bound state for small $\beta > 0$ expectedly depends on the profile function; cf. *Exner-Minakov-14, Pankrashkin-Popoff-16*.

(ii) Technique should be different, because BS-principle is not available.

Summary

Motivated by the open problem

$\sigma_d(\mathsf{H}_{\alpha,\Sigma}) \neq \varnothing$ for any $\Sigma \neq \Pi$ and all $\alpha > 0$?

The positive answer was known for suff. large $\alpha > 0$ *(Exner-Kondej-03)*.

Main results

- We prove $\sigma_d \neq \varnothing$ for any $\alpha > 0$ fixed and a small local deformation.
- For sufficiently small local deformation $\#\sigma_d = 1$.
- The lowest eigenvalue is asymptotically expanded in terms of the Fourier transform of the profile function.

Key features of the proof

- Involves applying BS-principle several times.
- Its by-product is an implicit scalar equation on the lowest eigenvalue.

Thank you

P. Exner, S. Kondej, and V. L., Asymptotics of the bound state induced by δ-interaction supported on a weakly deformed plane, arXiv:1703.10854.

Thank you for your attention!