Faber-Krahn inequalities for the Robin Laplacian on exterior domains

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The Faber-Krahn inequality

A bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, with the boundary $\partial\Omega$; ball $\mathcal{B} = \mathcal{B}_R \subset \mathbb{R}^d$

Dirichlet eigenvalues of the Laplacian on Ω

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad \Longrightarrow \quad 0 < \lambda_1^{\rm D}(\Omega) \leq \lambda_2^{\rm D}(\Omega) \leq \lambda_3^{\rm D}(\Omega) \leq \dots$$

The Faber-Krahn inequality (Faber-1923, Krahn-1926)

$$\begin{cases} |\Omega| = |\mathcal{B}| \\ \Omega \not\approx \mathcal{B} \end{cases} \quad \Longrightarrow \quad \boxed{\lambda_1^{\rm D}(\Omega) > \lambda_1^{\rm D}(\mathcal{B})}$$

For the Neumann Laplacian similar inequality is trivial because $\lambda_1^{\rm N}(\Omega) = 0$. It becomes non-trivial for the Robin Laplacian.

The original Faber-Krahn technique fails in the Robin case.

FK-inequality for the Robin Laplacian on a bounded domain

Robin eigenvalues of the Laplacian on Ω

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ \frac{\partial u}{\partial n} + \alpha u = 0, & \text{on } \partial\Omega, \end{cases} \quad \Longrightarrow \quad \lambda_1^\alpha(\Omega) \le \lambda_2^\alpha(\Omega) \le \lambda_3^\alpha(\Omega) \le \dots$$

$\frac{\partial u}{\partial n}$ – normal derivative with the outer normal n to Ω. $\alpha \in \mathbb{R}$ – coupling.

The Bossel-Daners inequality (Lip. Ω, $\boxed{\alpha > 0}$, Bossel-86, Daners-06)

$$\begin{cases} |\Omega| = |\mathcal{B}| \\ \Omega \not\cong \mathcal{B} \end{cases} \quad \Longrightarrow \quad \boxed{\lambda_1^\alpha(\Omega) > \lambda_1^\alpha(\mathcal{B})}$$

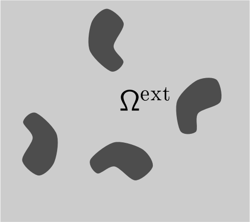
Flipped inequality (C^2-smooth Ω, $\boxed{\alpha < 0}$, Antunes-Freitas-Krejčiřík-16)

$$\begin{cases} |\partial\Omega| = |\partial\mathcal{B}| \\ \Omega, \mathcal{B} \subset \mathbb{R}^2 \end{cases} \quad \Longrightarrow \quad \boxed{\lambda_1^\alpha(\Omega) \le \lambda_1^\alpha(\mathcal{B})}$$

The Robin Laplacian on an exterior domain

Exterior domain

$\Omega^{\rm ext} := \mathbb{R}^d \setminus \overline{\Omega}$, where $\Omega \subset \mathbb{R}^d$ is a bounded domain, having $N_\Omega < \infty$ simply connected smooth components.



$\Omega^{\rm ext}$ (filled in gray) is connected ($N_\Omega = 4$).

$$Q_\alpha^{\Omega^{\rm ext}}[u] = \int_{\Omega^{\rm ext}} |\nabla u|^2 + \alpha \int_{\partial\Omega^{\rm ext}} |u|^2, \qquad \mathrm{dom}\, Q_\alpha^{\Omega^{\rm ext}} = H^1(\Omega^{\rm ext}).$$

The Robin Laplacian on $\Omega^{\rm ext}$

$Q_\alpha^{\Omega^{\rm ext}} \xrightarrow{1^{st}\text{-repr.}} -\Delta_\alpha^{\Omega^{\rm ext}}$ self-adjoint in $L^2(\Omega^{\rm ext})$.

$$-\Delta_\alpha^{\Omega^{\rm ext}} u = -\Delta u,$$

$$\mathrm{dom}\,(-\Delta_\alpha^{\Omega^{\rm ext}}) = \left\{u \in H^1(\Omega^{\rm ext}) \colon \Delta u \in L^2(\Omega^{\rm ext}), \frac{\partial u}{\partial n} = \alpha u \text{ on } \partial\Omega\right\}$$

Spectral shape optimisation for $-\Delta_\alpha^{\Omega^{\rm ext}}$

The Rayleigh quotient for the lowest spectral point of $-\Delta_\alpha^{\Omega^{\rm ext}}$

$$\lambda_1^\alpha(\Omega^{\rm ext}) = \inf_{\substack{u\in H^1(\Omega^{\rm ext})\\ u\neq 0}} \frac{Q_\alpha^{\Omega^{\rm ext}}[u]}{\|u\|^2_{L^2(\Omega^{\rm ext})}} = \inf \sigma(-\Delta_\alpha^{\Omega^{\rm ext}})\,.$$

Proposition

$$\begin{cases}\sigma_{\rm ess}(-\Delta_\alpha^{\Omega^{\rm ext}}) = [0,\infty) \\ \lambda_1^\alpha(\Omega^{\rm ext}) < 0 \ \ \textit{iff} \ \ \alpha < \alpha_\star(\Omega^{\rm ext})\end{cases}, \ \textit{where} \ \begin{cases}\alpha_\star(\Omega^{\rm ext}) = 0, & d = 2\\ \alpha_\star(\Omega^{\rm ext}) < 0, & d \geq 3.\end{cases}$$

Why spectral shape optimisation for $-\Delta_\alpha^{\Omega^{\rm ext}}$?

- New geometric setting: not much is known so far.
- Robin BC is crucial: for Dirichlet BC the problem is meaningless.
- Interplay with continuous spectrum: optimization of novel spectral quantities like $\alpha_\star(\Omega^{\rm ext})$.

Spectral isoperimetric inequality for exterior planar domains

Theorem (Krejčiřík-VL-17, $d = 2$, $\boxed{\alpha < 0}$)

$$\frac{|\partial\Omega|}{N_\Omega} = |\partial\mathcal{B}| \quad \Longrightarrow \quad \boxed{\lambda_1^\alpha(\Omega^{\rm ext}) \le \lambda_1^\alpha(\mathcal{B}^{\rm ext})}$$

Key tools for the proof

- Rayleigh quotient for $\lambda_1^\alpha(\Omega^{\rm ext})$ written in the parallel coordinates. Radial variable replaced by distance from $\partial\Omega$ (Payne-Weinberger-61).
- Radially symmetric ground-state of $-\Delta_\alpha^{\mathcal{B}^{\rm ext}}$ is transplanted from $\mathcal{B}^{\rm ext}$ onto $\Omega^{\rm ext}$.
- Min-max principle & total curvature identity $\int_{\partial\Omega} \kappa(s){\rm d}s = 2\pi N_\Omega$.

Corollary (Krejčiřík-VL-17, $d = 2$, $\boxed{\alpha < 0}$)

$$\begin{cases} |\partial\Omega| = |\partial\mathcal{B}| \ \ \textit{or} \ \ |\Omega| = |\mathcal{B}| \\ N_\Omega = 1 \end{cases} \quad \Longrightarrow \quad \boxed{\lambda_1^\alpha(\Omega^{\rm ext}) \le \lambda_1^\alpha(\mathcal{B}^{\rm ext})}$$

On the constraint $\frac{|\partial\Omega|}{N_\Omega} = |\partial\mathcal{B}|$

For $N = N_\Omega \geq 2$, it is impossible to replace the constraint

$$\frac{|\partial\Omega|}{N} = |\partial\mathcal{B}_R| \qquad \text{by} \qquad |\partial\Omega| = |\partial\mathcal{B}_R|\,.$$

Union of N disjoint disks

$\Omega = \cup_{n=1}^{N}\mathcal{B}_r(x_n)$ where $|x_n - x_m| > 2r$, $n \neq m$.

$|\partial\Omega| = |\partial\mathcal{B}_R| \Rightarrow r = \frac{R}{N}$

Strong coupling $\boxed{\alpha \to -\infty}$ (Pankrashkin-Popoff-16)

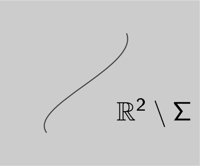
$\lambda_1^\alpha(\Omega^{\rm ext}) - \lambda_1^\alpha(\mathcal{B}_R^{\rm ext}) = |\alpha|\left(\frac{1}{r} - \frac{1}{R}\right) + o(\alpha) = |\alpha|\frac{N-1}{R} + o(\alpha).$

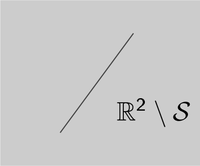
For sufficiently large $|\alpha|$

The inequality flips $\lambda_1^\alpha(\Omega^{\rm ext}) > \lambda_1^\alpha(\mathcal{B}_R^{\rm ext})$.

The Robin Laplacian on a plane with a cut

$\Sigma \subset \mathbb{R}^2$ – smooth open arc. $\mathcal{S} \subset \mathbb{R}^2$ – a line segment.





$$Q_\alpha^{\mathbb{R}^2\setminus\Sigma}[u] = \int_{\mathbb{R}^2} |\nabla u|^2 + \alpha \int_\Sigma (|\gamma_+ u|^2 + |\gamma_- u|^2), \quad \mathrm{dom}\, Q_\alpha^{\mathbb{R}^2\setminus\Sigma} = H^1(\mathbb{R}^2\setminus\Sigma).$$

The traces $\gamma_\pm u$ onto two faces of Σ need not be the same!

$-\Delta_\alpha^{\mathbb{R}^2\setminus\Sigma}$ and its lowest spectral point $\lambda_1^\alpha(\mathbb{R}^2 \setminus \Sigma)$ are defined similarly.

$\sigma_{\rm ess}(-\Delta_\alpha^{\mathbb{R}^2\setminus\Sigma}) = [0,\infty)$ and $\lambda_1^\alpha(\mathbb{R}^2 \setminus \Sigma) < 0$, $\forall \alpha < 0$.

Spectral isoperimetric inequality for the plane with a cut

Theorem (VL-16, $d = 2$, $\alpha < 0$)

$$\begin{cases} |\Sigma| = |\mathcal{S}| \\ \Sigma \not\cong \mathcal{S} \end{cases} \quad \Longrightarrow \quad \boxed{\lambda_1^\alpha(\mathbb{R}^2 \setminus \Sigma) < \lambda_1^\alpha(\mathbb{R}^2 \setminus \mathcal{S})}$$

Key tools for the proof

- Min-max principle.
- Birman-Schwinger principle (boundary integral reformulation).
- Line segment is the shortest path connecting two endpoints.

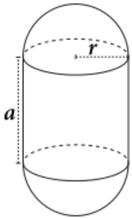
Inspired by the proof of isoperimetric inequality for the 1^{st}-eigenvalue of Schrödinger operator with δ-interaction on a loop (Exner-Harrell-Loss-06).

The constraint $|\partial\Omega| = |\partial\mathcal{B}|$ is "wrong" for $d \geq 3$

Long cylinder with 2 hemispherical caps

$\Omega_{r,a} = \mathrm{Conv}(\mathcal{B}_r(x_0) \cup \mathcal{B}_r(x_1))$, where $|x_0 - x_1| = a$.

$\Omega_\star \subset \mathbb{R}^d$ possesses a flat part of $\partial\Omega_\star$.



- $\forall$ suff. small $r \in (0, \frac{(d-2)R}{d-1})$: $\exists\, a > 0$ such that $|\partial\Omega_{r,a}| = |\partial\mathcal{B}_R|$.
- $|\partial\Omega_\star| = |\partial\mathcal{B}_R|$.

Strong coupling $\boxed{\alpha \to -\infty}$ **(Pankrashkin-Popoff-16)**

$$\lambda_1^\alpha(\Omega_{r,a}^{\mathrm{ext}}) - \lambda_1^\alpha(\mathcal{B}_R^{\mathrm{ext}}) = |\alpha| \left(\frac{d-2}{r} - \frac{d-1}{R}\right) + o(\alpha),$$
$$\lambda_1^\alpha(\Omega_\star^{\mathrm{ext}}) - \lambda_1^\alpha(\mathcal{B}_R^{\mathrm{ext}}) \leq -\frac{|\alpha|(d-1)}{R} + o(\alpha).$$

For all suff. large $|\alpha|$, $\boxed{\lambda_1^\alpha(\Omega_{r,a}^{\mathrm{ext}}) > \lambda_1^\alpha(\mathcal{B}_R^{\mathrm{ext}})}$ and $\boxed{\lambda_1^\alpha(\Omega_\star^{\mathrm{ext}}) < \lambda_1^\alpha(\mathcal{B}_R^{\mathrm{ext}})}$.

Curvatures

$\Omega \subset \mathbb{R}^d$, $d \geq 3$, a bounded smooth simply connected domain.

Principal curvatures of $\partial\Omega$

$\kappa_1, \kappa_2, \ldots, \kappa_{d-1}$ – eigenvalues of the Weingarten map, non-negative for convex Ω.

The mean curvature of $\partial\Omega$

$$M := \frac{\kappa_1 + \kappa_2 + \cdots + \kappa_{d-1}}{d-1}.$$

Averaged $(d-1)^{\mathrm{st}}$-power of the mean curvature

$$\mathcal{M}(\partial\Omega) = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} M^{d-1}(s)\mathrm{d}\sigma(s).$$

- For $d = 3$: $\int_{\partial\Omega} M^2(s)\mathrm{d}\sigma(s)$ is the famous Willmore energy of $\partial\Omega$.
- $\mathcal{M}(\partial\mathcal{B}_R) = R^{-(d-1)}$.

Spectral shape optimization for $d \geq 3$

$$\mathcal{M}(\partial\Omega) = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} M^{d-1}(s)\mathrm{d}\sigma(s)$$

Theorem (Krejčiřík-VL-17, $d \geq 3$, $\alpha < 0$)

$$\begin{cases} \mathcal{M}(\partial\Omega) = \mathcal{M}(\partial\mathcal{B}) \\ \Omega \text{ convex} \end{cases} \quad \Longrightarrow \quad \begin{cases} \lambda_1^\alpha(\Omega^{\rm ext}) \leq \lambda_1^\alpha(\mathcal{B}^{\rm ext}) \\ \alpha_\star(\Omega^{\rm ext}) \geq \alpha_\star(\mathcal{B}^{\rm ext}) \end{cases}$$

Key points

- Rayleigh quotient for $\lambda_1^\alpha(\Omega^{\rm ext})$ rewritten in parallel coordinates (for convex Ω the procedure simplifies).
- Transplantation of the ground-state for $-\Delta_\alpha^{\mathcal{B}^{\rm ext}}$.
- Geometric inequalities for convex bodies involved.
- Open problem: Is the result true for a class of non-convex Ω?

Summary

In the two-dimensional setting ($d = 2$, $\alpha < 0$)

⋆ $\lambda_1^\alpha(\Omega^{\rm ext}) \le \lambda_1^\alpha(\mathcal{B}^{\rm ext})$ for Ω having N_Ω bounded simply connected smooth components and satisfying $\frac{|\partial\Omega|}{N_\Omega} = |\partial\mathcal{B}|$.

⋆ $\lambda_1^\alpha(\Omega^{\rm ext}) \le \lambda_1^\alpha(\mathcal{B}^{\rm ext})$ for a smooth simply connected bounded domain Ω satisfying either $|\partial\Omega| = |\partial\mathcal{B}|$ or $|\Omega| = |\mathcal{B}|$.

⋆ $\lambda_1^\alpha(\mathbb{R}^2 \setminus \Sigma) \le \lambda_1^\alpha(\mathbb{R}^2 \setminus \mathcal{S})$ for a smooth arc Σ satisfying $|\Sigma| = |\mathcal{S}|$.

In the higher space dimensional setting ($d \ge 3$, $\alpha < 0$)

⋆ The constraint $|\partial\Omega| = |\partial\mathcal{B}|$ is "wrong" as a counterexample shows.

⋆ $\lambda_1^\alpha(\Omega^{\rm ext}) \le \lambda_1^\alpha(\mathcal{B}_R^{\rm ext})$ and $\alpha_\star(\Omega^{\rm ext}) \ge \alpha_\star(\mathcal{B}_R^{\rm ext})$ for a bounded smooth convex domain Ω satisfying $|\partial\Omega|^{-1}\int_{\partial\Omega} M^{d-1} = R^{-(d-1)}$.

Thank you

- D. Krejčiřík and V. L., Optimisation of the lowest Robin eigenvalue in the exterior of a compact set, to appear in J. Convex Anal., `arXiv:1608.04896`.
- D. Krejčiřík and V. L., Optimisation of the lowest Robin eigenvalue in the exterior of a compact set, II: non-convex domains and higher dimensions, `arXiv:1707.02269`.
- V. L., Spectral isoperimetric inequalities for δ-interactions on open arcs and for the Robin Laplacian on planes with slits, `arXiv:1609.07598`.

Thank you for your attention!