Optimization of lowest Robin eigenvalues on 2-manifolds and unbounded cones

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Outline

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1. Motivation & background
2. Optimization on 2-manifolds
3. Optimization on unbounded cones
4. Summary

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Purely discrete spectrum

$$\lambda_1^\beta(\Omega) \le \lambda_2^\beta(\Omega) \le \dots \le \lambda_k^\beta(\Omega) \le \dots \qquad (\beta \cdot \lambda_1^\beta(\Omega) \ge 0)$$

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2. maximizer $(\beta < 0,\ |\partial\Omega| = \mathrm{const},\ d = 2)$,
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1^{st} main objective

To generalize (2) for 2-manifolds.

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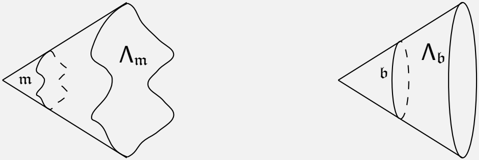

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$$\mathsf{H}_{\beta,\Lambda_{\mathfrak{m}}} \simeq \beta^2 \mathsf{H}_{-1,\Lambda_{\mathfrak{m}}} \qquad \Longrightarrow \qquad \mathsf{H}_{\Lambda_{\mathfrak{m}}} := \mathsf{H}_{-1,\Lambda_{\mathfrak{m}}}.$$

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Proposition (Pankrashkin-16)

- $\sigma_{\rm ess}(\mathsf{H}_{\Lambda_\mathfrak{m}}) = [-1, \infty)$.
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$2^{\rm nd}$ main objective

To obtain an isoperimetric inequality for $\lambda_1(\Lambda_{\mathfrak{m}})$.

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$\mathcal{N}_\circ$ – 2-manifold without boundary of constant Gauss curvature $K_\circ$.

$\mathcal{B}^\circ \subset \mathcal{N}_\circ$ – a geodesic disk.

$$\mathcal{N}_\circ \simeq \begin{cases} \mathbb{R}^2, & K_\circ = 0, \\ \mathbb{S}^2_{\frac{1}{\sqrt{K_\circ}}}, & K_\circ > 0. \end{cases}$$

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Theorem (`Khalile-L-19`)

$$\begin{cases} |\partial\mathcal{M}| = |\partial\mathcal{B}^\circ| \\ K_\circ \cdot \max\{|\mathcal{M}|, |\mathcal{B}^\circ|\} \leq 2\pi \end{cases} \implies \lambda_1^\beta(\mathcal{M}) \leq \lambda_1^\beta(\mathcal{B}^\circ), \qquad \forall\, \beta < 0.$$

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Counterexample on $\mathbb{S}^2$ ($K_\circ = 1$): $\mathcal{B}^\circ \subset \mathbb{S}^2$, $|\mathcal{B}^\circ| < 2\pi$, $\mathcal{M} := \mathbb{S}^2 \setminus \mathcal{B}^\circ$

Weak coupling expansions $\beta \to 0$

$$\lambda_1^\beta(\mathcal{M}) \sim \frac{\beta|\partial\mathcal{B}^\circ|}{4\pi - |\mathcal{B}^\circ|} \qquad \text{and} \qquad \lambda_1^\beta(\mathcal{B}^\circ) \sim \frac{\beta|\partial\mathcal{B}^\circ|}{|\mathcal{B}^\circ|}.$$

Hence, $\lambda_1^\beta(\mathcal{M}) > \lambda_1^\beta(\mathcal{B}^\circ)$ for $\beta < 0$ with $|\beta|$ small.

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Variational characterisation

$$\lambda_1^\beta(\mathcal{M}) = \inf_{u\in H^1(\mathcal{M})\setminus\{0\}} \frac{\|\nabla u\|^2_{L^2(\mathcal{M};\mathbb{C}^2)} + \beta\|u|_{\partial\mathcal{M}}\|^2_{L^2(\partial\mathcal{M})}}{\|u\|^2_{L^2(\mathcal{M})}}$$

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The ground-state $u_\circ(x) = \psi(\rho_{\partial\mathcal{B}^\circ}(x))\colon \mathcal{B}^\circ \to \mathbb{R}$ of $\mathsf{H}_{\beta,\mathcal{B}^\circ}$ is a radial function in geodesic polar coordinates on $\mathcal{B}^\circ$.

$$u_\star(x) := \psi\left(\rho_{\partial\mathcal{M}}(x)\right) \in H^1(\mathcal{M}) \qquad \text{(test function)}$$

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$\|u_\star|_{\partial\mathcal{M}}\|_{L^2(\partial\mathcal{M})} = \|u_\circ|_{\partial\mathcal{B}^\circ}\|_{L^2(\partial\mathcal{B}^\circ)}$ (trivial).

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$$\lambda_1^\beta(\mathcal{M}) \le \frac{\| \nabla u_\star \|^2_{L^2(\mathcal{M};\mathbb{C}^2)} + \beta \| u_\star|_{\partial\mathcal{M}} \|^2_{L^2(\partial\mathcal{M})}}{\| u_\star \|^2_{L^2(\mathcal{M})}}$$

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An application to the Steklov-type eigenvalues

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Proposition (Khalile-L-19)

$$\begin{cases} |\partial\mathcal{M}| = |\partial\mathcal{B}^\circ| \\ K_\circ \cdot \max\{|\mathcal{M}|, |\mathcal{B}^\circ|\} \leq 2\pi \end{cases} \Longrightarrow \quad \sigma_1^\lambda(\mathcal{M}) \leq \sigma_1^\lambda(\mathcal{B}^\circ), \qquad \forall\, \lambda < 0.$$

Outline

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$$H^1(\Lambda_\mathfrak{m}) \ni u \mapsto \int_{\Lambda_\mathfrak{m}} |\nabla u|^2 \mathrm{d}x - \int_{\partial\Lambda_\mathfrak{m}} |u|^2 \mathrm{d}\sigma \quad \Rightarrow \quad \mathsf{H}_{\Lambda_\mathfrak{m}} = \mathsf{H}^*_{\Lambda_\mathfrak{m}} \quad \text{in} \quad L^2(\Lambda_\mathfrak{m})$$

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An analogous result is known for δ-interactions on $\partial\Lambda_\mathfrak{m}$ (Exner-L-17). The method of the proof used there is not applicable any more.

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Problem II: optimization on cones

The Laplacian on an unbounded three-dimensional Euclidean cone + an attractive Robin BC.

Under a constraint of fixed perimeter of the cross-section, the lowest Robin eigenvalue is maximized by the circular cone.

Thank you

Thank you

M. Khalile and V. L., Spectral isoperimetric inequalities for Robin Laplacians on 2-manifolds and unbounded cones, arXiv:1909.10842.

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Thank you for your attention!